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CREEP BENDING AND BUCKLING OF LINEARLY VISCOELASTIC COLUMNS

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CREEP BENDING AND BUCKLING OF LINEARLY VISCOELASTIC COLUMNS

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SUMMARY

The general dynamic equation of creep bending of a beam loaded laterally and axially was derived for a linearly viscoelastic material whose mechanical properties can be characterized by four parameters. The material can exhibit instantaneous and retarded elasticity as well as pure flow.

The equation derived was used to obtain the creep bending deflection of a beam in pure bending and of a column with initial sinusoidal deviation from straightness. As expected, the ratio of the creep deflections of the beam in pure bending and the deflections of a corresponding purely elastic structure is identical to the ratio of the creep strain and the corresponding elastic strain of a bar under simple tension or compression.

The results of the analysis of the creep deflection of the column showed that the deflections increase continuously with time and become infinitely large only when the loading time is correspondingly large. However, large deflections are obtained in reasonably short periods of time if the applied load is near to the Euler load of the column. The deflection-time curves obtained from a numerical example are of the same type as those determined by experiment with aluminum columns.

INTRODUCTION

The problem of the determination of the behavior of beams and columns under conditions conducive to creep deformation has been given attention only in the past several years (see, for example, refs. 1 to 8). This problem is becoming increasingly important because of the high temperatures at which high-speed missiles and aircraft as well as modern power plants operate. It is the purpose of the present report to investigate analytically the nature of the creep deformation of beams formed from materials which can be assumed to exhibit linearly viscoelastic properties either at room or elevated temperatures.

The fundamental behavior of ideal viscoelastic materials when subjected to various types of loading has been discussed rather fully in the literature (see, for example, refs. 2 and 9). In this work the viscoelastic material is represented by a system of linearly elastic elements (such as Hookean springs) and linearly viscous elements (Newtonian dashpots) connected in various series and parallel configurations (see figs. 1(a) and 1(b)). An important class of problems which may be analyzed with the aid of models of the aforementioned type consists of those involving the effect of viscoelastic creep on the response of beams and columns to various types of loads. Freudenthal (refs. 1 and 2) has discussed such problems in which the structural material is considered to be represented by a single Maxwell element consisting of a linear spring connected in series with a linear dashpot; hence the material properties are defined by two parameters (fig. 1(c)). However, it is shown in reference 9 that in order to approximate real materials, such as amorphous linear polymers, by a model consisting of springs and dashpots several such elements must be incorporated in the model. Thus it is the purpose of the present report to extend the application of the concept of representing the mechanical properties of viscoelastic beams and columns by networks of springs and dashpots to models characterized by more than two parameters. In particular, the general dynamic equation of motion of a bent beam column is derived for a material represented by a Maxwell element connected in series with a Kelvin or Voigt element where the latter element consists of a spring and dashpot in parallel (fig. 1(d)), and hence the complete model involves four parameters (fig. 2). Such a model is capable of exhibiting instantaneous and retarded elastic response, as well as pure viscous flow.

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SYMBOLS

A	cross-sectional area
A,B,C_1,C_2	constants of integration
A',B'	constants
a	amplitude of function representing initial deviation from straightness
E_1,E_2	moduli of elasticity
G	function of time

I	moment of inertia of cross-sectional area of beam or column with respect to a principal centroidal axis
$k = 1 + (\tau_2/\tau_1) + (E_1/E_2)$	
L	length of beam or column
M	bending moment
$\bar{M}$	bending moment due to lateral and inertial loads
M_o	constant bending moment
m	mass per unit length
n_1, n_2	constants
P	axially compressive end load
P_E	Euler load, $\pi^2 E_1 I/L^2$
p	applied lateral load per unit length
T	function of time
t	time
w	deflection due to loads
$w_{elastic}$	elastic deflection due to loads
w_i	function representing initial deviation from straightness
x	coordinate along centroidal axis of beam or column
y	principal axis of cross-sectional area of beam or column
z	distance from xy-plane
α	ratio of end load and Euler load
ϵ	strain, positive in tension
$\epsilon_{elastic}$	elastic strain
ϵ_o	strain at $z = 0$

$\epsilon_1, \epsilon_2, \epsilon_{1a}, \epsilon_{1b}$ strain contributions

λ_1, λ_2 coefficients of viscosity

ρ radius of curvature

σ stress, positive in tension

σ_{2a}, σ_{2b} stresses

τ_1, τ_2 relaxation and retardation times, respectively

$(\dot{}) \equiv \partial(\ )/\partial t$

$(\ )_x \equiv \partial(\ )/\partial x$

DERIVATION OF DIFFERENTIAL EQUATION OF BENDING

Basic Theory

The mechanical behavior of viscoelastic materials, such as amorphous linear polymers, under conditions of small stresses and strain, has been discussed in great detail by Alfrey in reference 9. It is shown in this reference that an analogy exists between the mechanical behavior of such materials and the mechanical behavior of Hookean springs (linear elasticity) and Newtonian dashpots (linear viscosity) connected in various series and parallel networks (fig. 1). If two or more elements are coupled in series producing a configuration such as the Maxwell model of figure 1(c) each of the elements must support the total load applied to the model, and the total extension of the system is the sum of the individual extensions of the elements involved. On the other hand, if elements are coupled in parallel (such being the case with the Kelvin or Voigt model of fig. 1(d)), the total load supported by the model is equal to the sum of the loads carried by the individual elements, whereas the total extension of the model is equal to the extension of each of its components. The Maxwell model is capable of exhibiting instantaneous elastic response and viscous flow when subjected to arbitrary loading conditions. The former action produces recoverable deformation, whereas the latter action produces permanent deformation. The phenomenon of stress relaxation can be represented with the aid of such a Maxwell model. The Voigt model represents a retarded or time-dependent elastic response to applied loads and is capable of exhibiting elastic aftereffects. It is shown in reference 9 that the mechanical behavior of combinations of Voigt models in series, such as that shown in figure 1(a) where the first and last elements may be considered as degenerate Voigt elements, or of combinations of Maxwell

elements in parallel (fig. 1(b)) is analogous to the mechanical behavior of amorphous linear polymers under conditions of small stresses and strains. In order to determine the pertinent elasticity and viscosity coefficients to be used to complete the choice of a model representing a given material, suitable creep or relaxation tests must be undertaken. The various coefficients are then determined empirically. Once these coefficients are known as functions of temperature (and they will be functions only of temperature if the material being represented is viscoelastic and the model chosen is of sufficiently complete structure), then the behavior under arbitrary conditions of loading and temperature of any isotropic, homogeneous structural member formed from the viscoelastic material can be described analytically. Although only those structural members which can be considered to be uniaxially stressed are considered herein, a generalized three-dimensional theory exists and is described in reference 9. However, the theory presented therein is valid only for models characterized by two parameters.

In the ensuing analysis the behavior of viscoelastic beams and columns is studied by assuming that the material of construction can be represented adequately by a network consisting of a spring in series with a Voigt element and a dashpot (see fig. 2). The model chosen can be considered as representing a first approximation to a more general model (see fig. 1(a)) and is capable of exhibiting instantaneous and retarded elastic response as well as viscous flow. Any model of simpler nature than that chosen is in general incapable of responding to loads in a manner analogous to known viscoelastic materials. In the present study the structures considered are assumed to be in a state of uniform constant temperature and hence the elasticity and viscosity coefficients are constants.

STRESS-STRAIN-TIME LAW

With the use of the principles governing the behavior of springs and dashpots coupled in series or parallel stated earlier, the general stress-strain-time relationship governing the behavior of the four-parameter model of figure 2 can be determined. If at any time t the stress σ is applied to the model as shown in figure 2, where σ is positive in tension, the strain contribution ϵ_{1a} of the elastic element (1a), having an elastic modulus E_1, is given by Hooke's law as

$$\epsilon_{1a} = \sigma/E_1 \tag{1}$$

The relation between the applied stress and corresponding strain in the viscous element (1b) is given by Newton's law of viscosity as

$$\dot{\epsilon}_{1b} = \sigma/\lambda_1 \tag{2}$$

in which λ_1 is a coefficient of viscosity and the dot above the strain indicates differentiation with respect to time. Thus the combined strain contribution ϵ_1 of elements (1a) and (1b) is related to the applied stress as follows:

$$\dot{\epsilon}_1 = \left(\dot{\sigma}/E_1\right) + \left(\sigma/\lambda_1\right) \tag{3}$$

Equation (3) is the well-known stress-strain-time relationship for a Maxwell model.

The strain contribution ϵ_2 of elements (2a) and (2b), respectively, is related to the applied stresses σ_{2a} and σ_{2b}, where the sum of these stresses is the applied stress σ, by expressions similar to equations (1) and (2). Thus

$$\epsilon_2 = \sigma_{2a}/E_2 \tag{4}$$

$$\dot{\epsilon}_2 = \sigma_{2b}/\lambda_2 \tag{5}$$

in which E_2 is the modulus of elasticity and λ_2 the coefficient of viscosity of the elastic and viscous elements (2a) and (2b), respectively.

Hence the strain ϵ_2 is related to the applied stress by the equation

$$\lambda_2\dot{\epsilon}_2 + E_2\epsilon_2 = \sigma \tag{6}$$

Equation (6) is the defining equation of a Kelvin or Voigt model.

The total strain in the model of figure 2 is

$$\epsilon = \epsilon_1 + \epsilon_2 \tag{7}$$

in which ϵ_1 and ϵ_2 are defined by equations (3) and (6), respectively. Elimination of ϵ_1 and ϵ_2 from equations (3), (6), and (7) leads to the following stress-strain-time relation governing the mechanical behavior of the four-parameter model considered:

$$\ddot{\sigma} + \left(k/\tau_2\right)\dot{\sigma} + \left(1/\tau_1\tau_2\right)\sigma = E_1\left[\ddot{\epsilon} + \left(1/\tau_2\right)\dot{\epsilon}\right] \tag{8}$$

in which $\tau_1 = \lambda_1/E_1$, the relaxation time of a Maxwell element with elasticity and viscosity coefficients E_1 and λ_1, respectively; $\tau_2 = \lambda_2/E_2$,

the retardation time of a Voigt element with elasticity and viscosity coefficients E_2 and λ_2, respectively; and $k = 1 + \left(\tau_2/\tau_1\right) + \left(E_1/E_2\right)$.

Equation (8) is thus the uniaxial stress-strain-time relationship for a viscoelastic material whose response to loading at a constant temperature is analogous to the behavior of the model shown in figure 2. The determination of the strain-time relation for a member under uniform tensile or compressive stress is facilitated by the direct application of the component equations (3) and (6) rather than the general equation (8). Thus equations (3) and (6) together with equation (7) and the condition that at $t = 0$ ϵ is completely elastic and equal to ϵ_1 result in the following strain-time relation (see ref. 9):

$$\epsilon/\epsilon_{elastic} = \left(E_1/E_2\right)\left(1 - e^{-t/\tau_2}\right) + \left(t/\tau_1\right) + 1 \qquad (8a)$$

in which the elastic strain $\epsilon_{elastic} = \sigma/E_1$. This relation is plotted in figure 3 for values of the elasticity and viscosity coefficients used in subsequent calculations.

Strain-Deflection Relation

Consider a beam loaded in a manner such that the bending moment vector M acting on any cross section coincides with a principal centroidal axis such as the y-axis in figure 4. With the assumption that cross sections of the undeformed beam remain plane and normal to the centroidal axis after bending, each element of the beam deforms under the influence of the bending moment and an axial load as indicated in figure 5. In this figure x is the centroidal axis of the beam; z is the distance from the xy-plane to any other parallel plane; ϵ_o and ϵ are the strains, shown positive in tension, at the xy-plane and at a distance z from this plane, respectively; and ρ is the radius of curvature of the centroidal axis. Thus the strain corresponding to any z is

$$\epsilon = \left(1 + \epsilon_o\right)(z/\rho) + \epsilon_o \qquad (9)$$

If the strains are considered small, equation (9) becomes

$$\epsilon = (z/\rho) + \epsilon_o \qquad (10)$$

Since the strain ϵ is a linear function of z, the stress is also a linear function of z (see eq. (8)). Hence, the stress distribution

across a section can be decomposed into a uniform stress and a linearly distributed stress which vanishes at the xy-plane. If a constant axially compressive load P is applied to the beam of cross-sectional area A, the uniform stress (and hence the stress at $z = 0$) is $-P/A$. Therefore, ϵ_o, which is caused by the constant stress $-P/A$, can be determined from equation (8a), in which $\epsilon_{elastic}$ is now taken as $-P/AE_1$.

For small displacements w of the centroidal axis

$$1/\rho = -w_{xx} \tag{11}$$

where a subscript x denotes one differentiation with respect to x. Thus for small strain and displacements

$$\epsilon = -zw_{xx} + \epsilon_o \tag{12}$$

Differential Equation of Bending of a Beam Column

With the aid of equations (8) and (10), the differential equation of bending is readily derived. Multiplication of equation (8) by $z\ dA$ and integration of the resulting equation over the area A yield

$$\int_A \ddot{\sigma} z\ dA + \left(k/\tau_2\right)\int_A \dot{\sigma} z\ dA + \left(1/\tau_1\tau_2\right)\int_A \sigma z\ dA =$$

$$E_1\int_A \ddot{\epsilon} z\ dA + \left(E_1/\tau_2\right)\int_A \dot{\epsilon} z\ dA \tag{13}$$

The moment M is related to the stress by the relation

$$M = \int_A \sigma z\ dA \tag{14}$$

Substitution of equations (10) and (14) into equation (13) together with

$$\int_A z^2\ dA = I$$

and

$$\int_A z\ dA = 0$$

where I is the moment of inertia of the cross-sectional area A with respect to the centroidal principal axis y (fig. 4), yields

$$E_1 I\left[\partial^2(1/\rho)/\partial t^2 + (1/\tau_2)\partial(1/\rho)/\partial t\right] = \ddot{M} + (k/\tau_2)\dot{M} + (1/\tau_1\tau_2)M \qquad (15)$$

Equation (15) is the general equation of bending of a beam of viscoelastic material whose mechanical analog is the model shown in figure 2. For small displacements the curvature $1/\rho$ is defined by equation (11) and hence equation (15) reduces to

$$-E_1 I\left[\ddot{w}_{xx} + (1/\tau_2)\dot{w}_{xx}\right] = \ddot{M} + (k/\tau_2)\dot{M} + (1/\tau_1\tau_2)M \qquad (16)$$

The bending moment can be expressed as

$$M = P(w + w_i) + \bar{M}(x,t) \qquad (17)$$

in which w_i represents an initial deviation from straightness of the original beam axis and $\bar{M}$ is the moment contribution due to applied lateral and inertial loads. Substitution of M from equation (17) into equation (16) yields

$$E_1 I\left[\ddot{w}_{xx} + (1/\tau_2)\dot{w}_{xx}\right] + P\left[\ddot{w} + (k/\tau_2)\dot{w} + (1/\tau_1\tau_2)w\right] =$$
$$-(P/\tau_1\tau_2)w_i - \ddot{\bar{M}} - (k/\tau_2)\dot{\bar{M}} - (1/\tau_1\tau_2)\bar{M} \qquad (18)$$

Furthermore, if m is the mass per unit length of the beam and P is an applied load per unit length of the beam, where m may vary with x and p with x and t, then

$$\bar{M}_{xx} = -p + m\ddot{w} \qquad (19)$$

Hence equation (18) becomes

$$\left\{E_1 I\left[\ddot{w}_{xx} + (1/\tau_2)\dot{w}_{xx}\right]\right\}_{xx} + \left\{P\left[\ddot{w} + (k/\tau_2)\dot{w} + (1/\tau_1\tau_2)w\right]\right\}_{xx} +$$
$$m\left[\ddot{\dot{w}} + (k/\tau_2)\ddot{w} + (1/\tau_1\tau_2)\ddot{w}\right] = -\left[(P/\tau_1\tau_2)w_i\right]_{xx} + \ddot{p} +$$
$$(k/\tau_2)\dot{p} + (1/\tau_1\tau_2)p \qquad (20)$$

Equation (20) is the general dynamic equation of bending with small deflection of a four-parameter viscoelastic material. If τ_2 is considered infinitely large, that is, the Voigt element of figure 2 is frozen in, then equation (20) reduces to the dynamic equation of bending of a beam of Maxwell material (see ref. 10).

INITIAL CONDITIONS

Regardless of the exact nature of any particular beam problem, certain initial conditions which must be imposed on all solutions of equation (20) are assumed in the present application of the viscoelastic model of figure 2. These conditions are that all external loads are applied instantaneously and hence the initial response is purely elastic (see ref. 9). Thus at $t = 0$

$$\epsilon = \epsilon_{1a} = \sigma/E_1 \tag{21}$$

and

$$\epsilon_{1b} = \epsilon_2 = 0 \tag{22}$$

Since $\epsilon = \epsilon_{1a} + \epsilon_{1b} + \epsilon_2$, equations (21) and (22) yield only two independent relations. Differentiation of equation (7) with respect to t and subsequent substitution of equations (3) and (6) into the resulting equation yield

$$\epsilon_2 = \left(\tau_2/E_1\right)\dot{\sigma} + \left[\left(k - 1\right)/E_1\right]\sigma - \tau_2\dot{\epsilon} \tag{23}$$

Introduction of equation (12) into equation (23) results in the following relation for ϵ_2:

$$\epsilon_2 = \left(\tau_2/E_1\right)\dot{\sigma} + \left[\left(k - 1\right)/E_1\right]\sigma + \tau_2\left(z\dot{w}_{xx} - \dot{\epsilon}_o\right) \tag{24}$$

Multiplication of equation (24) by $z\,dA$, integration over the area A, and introduction of equation (14) into the relation obtained yield for $\epsilon_2 = 0$ at $t = 0$

$$\left(\tau_2/E_1\right)\dot{M} + \left[\left(k - 1\right)/E_1\right]M + \tau_2 I\dot{w}_{xx} = 0 \tag{25}$$

Hence equations (21) and (25) are initial conditions which must be imposed on all solutions of equation (16) or (20).

APPLICATIONS OF DIFFERENTIAL EQUATION OF BENDING

Beam Under Pure Bending

Equation (16) can be readily applied to the problem of creep due to pure bending of a viscoelastic beam with inertial effects neglected. If the applied constant bending moment is designated as M_o, then equation (16) becomes

$$E_1 I\left[\ddot{w}_{xx} + \left(1/\tau_2\right)\dot{w}_{xx}\right] = \left(1/\tau_1\tau_2\right)M_o \tag{26}$$

Integration of this equation twice with respect to x and introduction of the conditions that, at $x = 0$ and L, $w = \dot{w} = 0$ yield

$$E_1 I\left[\ddot{w} + \left(1/\tau_2\right)\dot{w}\right] = \left(M_o/2\tau_1\tau_2\right)\left(Lx - x^2\right) \tag{27}$$

in which L is the length of the beam.

If w is written as

$$w = G(t)\left(M_o/2\tau_1\tau_2\right)\left(1/E_1 I\right)\left(Lx - x^2\right) \tag{28}$$

where $G(t)$ is a function of time only, then equation (27) reduces to

$$\ddot{G} + \left(1/\tau_2\right)\dot{G} = 1 \tag{29}$$

The general solution of equation (29) is

$$G = C_1 e^{-\left(t/\tau_2\right)} + \tau_2\left(t + C_2 - \tau_2\right) \tag{30}$$

where C_1 and C_2 are constants of integration. Thus equation (28) becomes

$$w = \left(M_o/2\tau_1\tau_2\right)\left(1/E_1 I\right)\left(Lx - x^2\right)\left[C_1 e^{-\left(t/\tau_2\right)} + \tau_2\left(t + C_2 - \tau_2\right)\right] \tag{31}$$

At $t = 0$, the initial deflections are purely elastic and hence

$$w = w_{elastic} = \left(M_o/2E_1 I\right)\left(Lx - x^2\right) \tag{32}$$

Also at $t = 0$, $\epsilon_2 = 0$ and hence equation (25) together with equation (31) yields

$$C_1 = -\left(E_1/E_2\right)\tau_1\tau_2 \tag{33}$$

Application of equation (32) and substitution of equation (33) into equation (31) result in the following value for C_2:

$$C_2 = \tau_1 k \tag{34}$$

Hence from equations (31), (33), and (34)

$$w = \left[\left(E_1/E_2\right)\left(1 - e^{-t/\tau_2}\right) + \left(t/\tau_1\right) + 1\right]\left(M_o/2E_1I\right)\left(Lx - x^2\right) \qquad (35)$$

In terms of the elastic displacement, equation (35) becomes

$$w/w_{elastic} = \left(E_1/E_2\right)\left(1 - e^{-t/\tau_2}\right) + \left(t/\tau_1\right) + 1 \qquad (36)$$

Equation (35) or (36) describes the pure-bending creep behavior of a viscoelastic beam whose material has a mechanical behavior under stress analogous to the behavior of the model of figure 2. Equation (36) is plotted in figure 3 and yields a creep curve identical to the creep curve for the same material under simple tensile or compressive stresses (see eq. (8a)).

Column With Initial Curvature

The behavior of a viscoelastic column whose centroidal axis initially deviates from a straight line can also be analyzed with the aid of equation (20). With inertial effects neglected equation (20) when applied to the simply supported uniform column of length L indicated in figure 6 reduces to

$$E_1I\left[\ddot{w}_{xxxx} + \left(1/\tau_2\right)\dot{w}_{xxxx}\right] + P\left[\ddot{w}_{xx} + \left(k/\tau_2\right)\dot{w}_{xx} + \left(1/\tau_1\tau_2\right)w_{xx}\right] = -\left(P/\tau_1\tau_2\right)w_{i_{xx}} \qquad (37)$$

If the initial deviation from straightness is taken as

$$w_i = a \sin(\pi x/L) \qquad (38)$$

where a is a constant, then the solution of equation (37) for the additional deflection w can be assumed in the form

$$w = aT(t) \sin(\pi x/L) \qquad (39)$$

where T(t) is a function of time only. Upon substitution of equations (38) and (39) into equation (37) the following equation results for the function T:

$$\ddot{T} + (1/\tau_2)\left\{\frac{\left[1 - k(P/P_E)\right]}{\left[1 - (P/P_E)\right]}\right\}\dot{T} - (1/\tau_1\tau_2)\left\{(P/P_E)\Big/\left[1 - (P/P_E)\right]\right\}T =$$

$$(1/\tau_1\tau_2)(P/P_E)\Big/\left[1 - (P/P_E)\right] \tag{40}$$

in which the Euler load $P_E = \pi^2 E_1 I/L^2$. The general solution of equation (40) is

$$T = Ae^{n_1 t} + Be^{n_2 t} - 1 \tag{41}$$

in which A and B are constants of integration and

$$\left.\begin{aligned} n_1 &= -\left[(1/2\tau_2)(1 - k\alpha)\Big/(1 - \alpha)\right] + \left\{\left[(1/2\tau_2)(1 - k\alpha)\Big/(1 - \alpha)\right]^2 + \right. \\ &\qquad \left.\left[(1/\tau_1\tau_2)\alpha\Big/(1 - \alpha)\right]\right\}^{1/2} \\ n_2 &= -\left[(1/2\tau_2)(1 - k\alpha)\Big/(1 - \alpha)\right] - \left\{\left[(1/2\tau_2)(1 - k\alpha)\Big/(1 - \alpha)\right]^2 + \right. \\ &\qquad \left.\left[(1/\tau_1\tau_2)\alpha\Big/(1 - \alpha)\right]\right\}^{1/2} \end{aligned}\right\} \tag{42}$$

where $\alpha = P/P_E$: Since α must be less than unity in order for the column to be initially stable, n_1 is real and positive and n_2 is real and negative.

Upon substitution of equation (41) into equation (39), the deflection w becomes

$$w = \left(Ae^{n_1 t} + Be^{n_2 t} - 1\right)a \sin(\pi x/L) \tag{43}$$

The initial conditions which must be imposed on equation (43) are (see eqs. (21) and (25))

$$t = 0, \qquad w = w_{elastic} = \left[\alpha/(1 - \alpha)\right]a \sin(\pi x/L) \tag{44}$$

$$t = 0, \qquad (\tau_2/E_1)\dot{M} + \left[(k - 1)/E_1\right]M + \tau_2 I\dot{w}_{xx} = 0 \tag{45}$$

Since $M = P(w + w_i)$, equation (45) together with equation (43) yields at $t = 0$

$$\left\{n_1 - \left[(k - 1)/\tau_2\right]\alpha/(1 - \alpha)\right\} A + \left\{ n_2 - \left[(k - 1)/\tau_2\right]\alpha/(1 - \alpha)\right\} B = 0 \quad (46)$$

Similarly, equations (43) and (44) yield

$$A + B = 1/(1 - \alpha) \quad (47)$$

Thus the constants A and B are

$$\left.\begin{aligned} A &= \left[1/(1 - \alpha)\right]\left\{ \left[(k - 1)/\tau_2\right]\left[\alpha/(1 - \alpha)\right] - n_2\right\}\Big/(n_1 - n_2) \\ B &= -\left[1/(1 - \alpha)\right]\left\{ \left[(k - 1)/\tau_2\right]\left[\alpha/(1 - \alpha)\right] - n_1\right\}\Big/(n_1 - n_2) \end{aligned}\right\} \quad (48)$$

Hence the solution can be expressed in the following nondimensional form:

$$(w + w_i)/(w_{elastic} + w_i) = A'e^{n_1 t} + B'e^{n_2 t} \quad (49)$$

in which

$$\left.\begin{aligned} A' &= \left\{\left[(k - 1)/\tau_2\right]\left[\alpha/(1 - \alpha)\right] - n_2\right\}\Big/(n_1 - n_2) \\ B' &= -\left\{ \left[(k - 1)/\tau_2\right]\left[\alpha/(1 - \alpha)\right] - n_1\right\}\Big/(n_1 - n_2) \end{aligned}\right\} \quad (50)$$

The ratio of the strain ϵ at any distance z from the xy-plane (fig. 5) and the elastic strain at $z = 0$, $\epsilon_{o_{elastic}}$, can now be determined from equation (12) together with equations (8a), (38), (44), and (49). Thus

$$\epsilon/\epsilon_{o_{elastic}} = -z(aA/I)(1/\alpha)\left\{\left[1/(1 - \alpha)\right]\left(A'e^{n_1 t} + B'e^{n_2 t}\right) - 1\right\}\sin(\pi x/L) + (E_1/E_2)\left(1 - e^{-t/\tau_2}\right) + (t/\tau_1) + 1 \quad (51)$$

in which $\epsilon_{o_{elastic}} = -P/AE_1$.

Equation (43) or (49) shows that for the column considered the deflections due to linearly viscous creep increase with time and approach infinity only as t approaches infinity for $\alpha = P/P_E < 1$. Equation (49) is plotted in figure 7 for various values of P/P_E for a material whose elasticity and viscosity coefficients are given in the figure. The curves show that for $P/P_E = 0.8$ the deflections increase much more rapidly with time than for the smaller values of P/P_E. For the same material described in figure 7, equation (51) is plotted in figure 8 for $x = L/2$, $P/P_E = 0.5$, and $z(aA/I) = \pm 0.1$. The curves show that on the compression (concave) side of the beam the compressive strain increases monotonically with time, while on the tension (convex) side the strain reaches a maximum value in compression, whereupon it decreases monotonically through zero, finally becoming tensile.

It may be noted that equation (49) reduces to the corresponding solution for a column constructed of a Maxwell element if τ_2 is taken infinitely large (see refs. 1 and 2).

DISCUSSION

The differential equation of bending of a beam or column was derived for a viscoelastic material whose mechanical behavior is analogous to that of the model shown in figure 2. This model, whose properties are defined by two elasticity and two viscosity coefficients, is the simplest model exhibiting instantaneous and retarded elasticity as well as pure flow. For materials whose properties are defined by more general linearly viscoelastic models (see fig. 1(a)) corresponding differential equations of bending can be derived in a manner similar to the derivation presented earlier in the text.

The results of the analysis of the creep deflections of a viscoelastic beam under pure bending are shown in figure 3. It is seen from equations (8a) and (36) that the ratio of the deflections of the beam and the deflections of the corresponding purely elastic structure at any time is identical to the ratio of the total strain and the purely elastic strain of a bar under uniaxial tensile or compressive stresses. Consideration of the linear nature of the stress-strain-time relation (eq. (8)) also leads to the conclusion that for any bent bar the stresses are linearly distributed across the depth since the strains have been so assumed. Hence for all cases of bending of a bar by static lateral loads the initial linear distribution of stresses across the depth of the beam remains unchanged and can be determined by consideration of equilibrium only, even though the deflections increase continuously with time.

The analysis of the creep deflections of a simply supported column with an initial sinusoidal deviation from straightness resulted in the observations that the deflections tend to be infinitely large with time and that for the applied load less than the Euler load the deflections remain finite for finite time (see eq. (49) and fig. 7). Because of the linear nature of the general differential equation of bending (eq. (20)) these observations are applicable to column problems similar to the one considered herein but with initial shapes other than the single sine wave. In the case of columns, as with beams, the stresses are linearly distributed across the depth (see eqs. (8) and (12)). However, the actual distribution changes continuously with time since the applied moment which is proportional to the total deflection changes with time.

The variation of strain with time at the midspan of the column at two points symmetrically located with respect to a plane normal to the principal plane of bending and containing the beam axis is shown in figure 8. The compressive strain on the compression side of the beam increases monotonically with time; whereas the strain on the tension side reaches a maximum value in compression, then decreases monotonically through zero, finally becoming tensile.

The deflection-time curves shown in figure 7 indicate that the deflections may increase very rapidly with time for end loads near the Euler load. It is interesting to note that the curves exhibit the primary and secondary creep characteristics observed in experiments with aluminum columns reported in references 6 and 8.

Polytechnic Institute of Brooklyn,
Brooklyn, N. Y., April 18, 1952.

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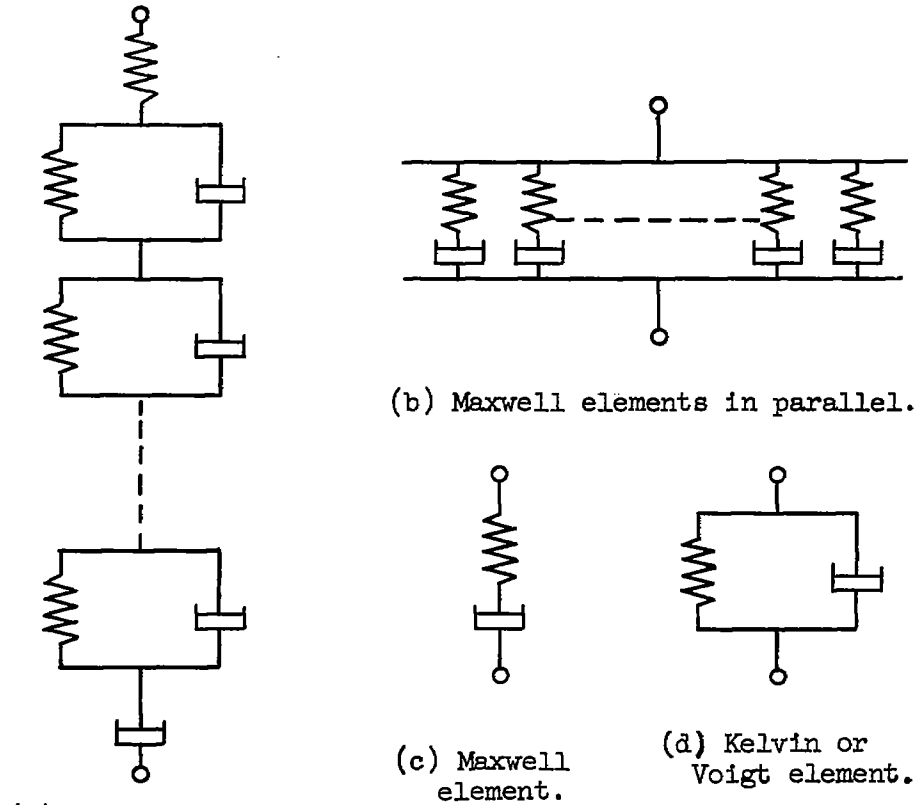

(a) Voigt elements in series.

(b) Maxwell elements in parallel.

(c) Maxwell element.

(d) Kelvin or Voigt element.

Figure 1.- Models used for study of mechanical behavior of linearly viscoelastic materials.

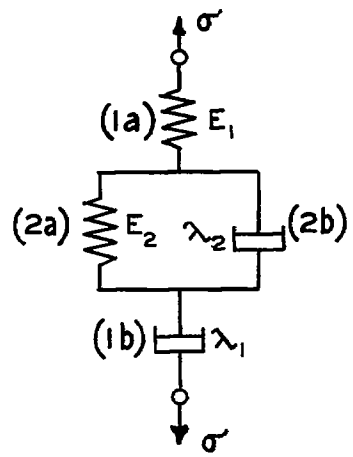

Figure 2.- Four-parameter viscoelastic model.

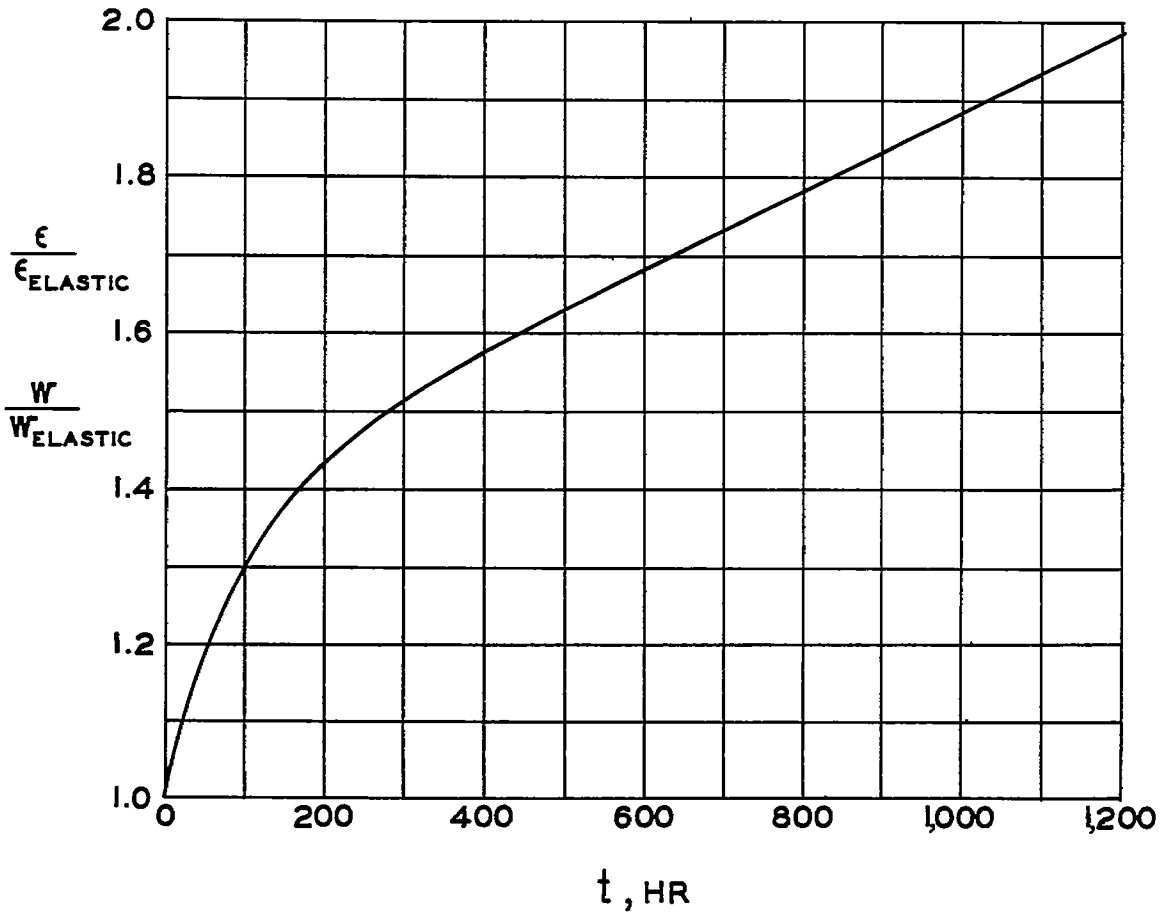

Figure 3.- Creep strain for uniform tension or compression and creep deflections for pure bending. $E_1 = 5 \times 10^5$ psi; $E_2 = 13 \times 10^5$ psi; $\lambda_1 = 10^9$ lb-hr/sq in.; $\lambda_2 = 13 \times 10^7$ lb-hr/sq in.; $\tau_1 = 2{,}000$ hr; $\tau_2 = 100$ hr.

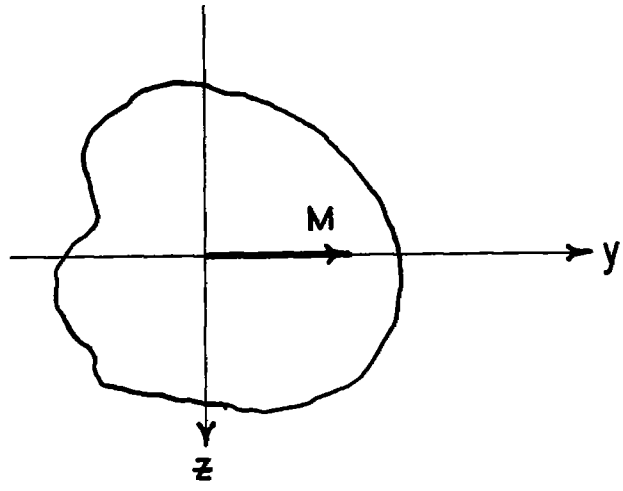

Figure 4.- Cross section of beam.

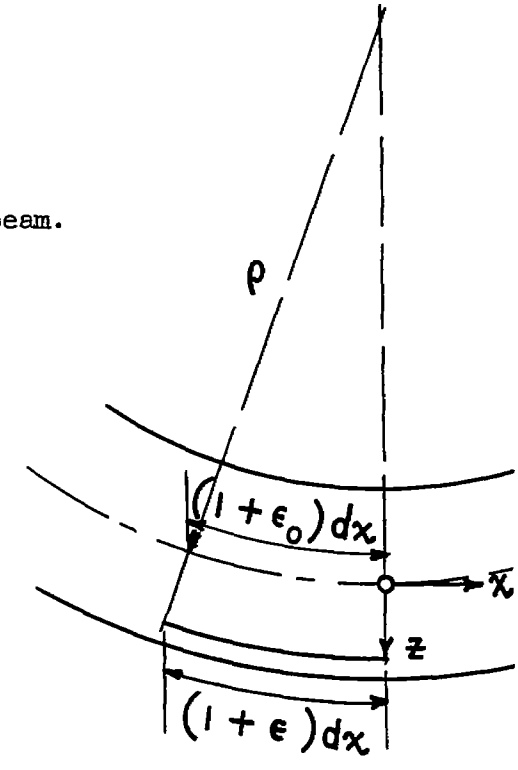

Figure 5.- Geometry of strains in deformed beam.

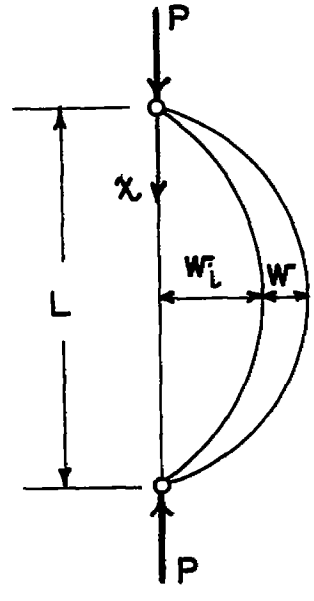

Figure 6.- Deflections of a simply supported initially curved column.

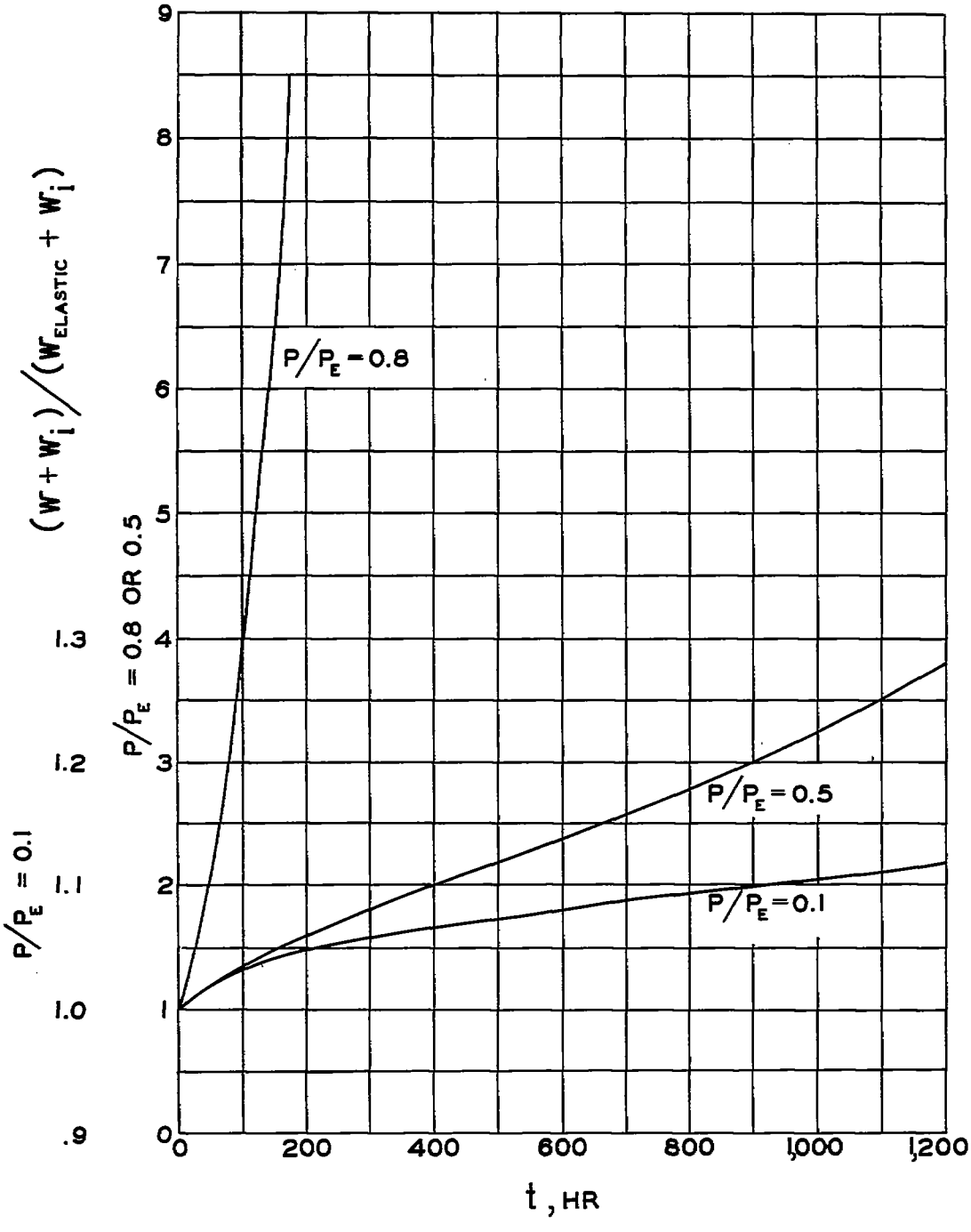

Figure 7.- Creep deflections of a simply supported initially curved column. $E_1 = 5 \times 10^5$ psi; $E_2 = 13 \times 10^5$ psi; $\lambda_1 = 10^9$ lb-hr/sq in.; $\lambda_2 = 13 \times 10^7$ lb-hr/sq in.

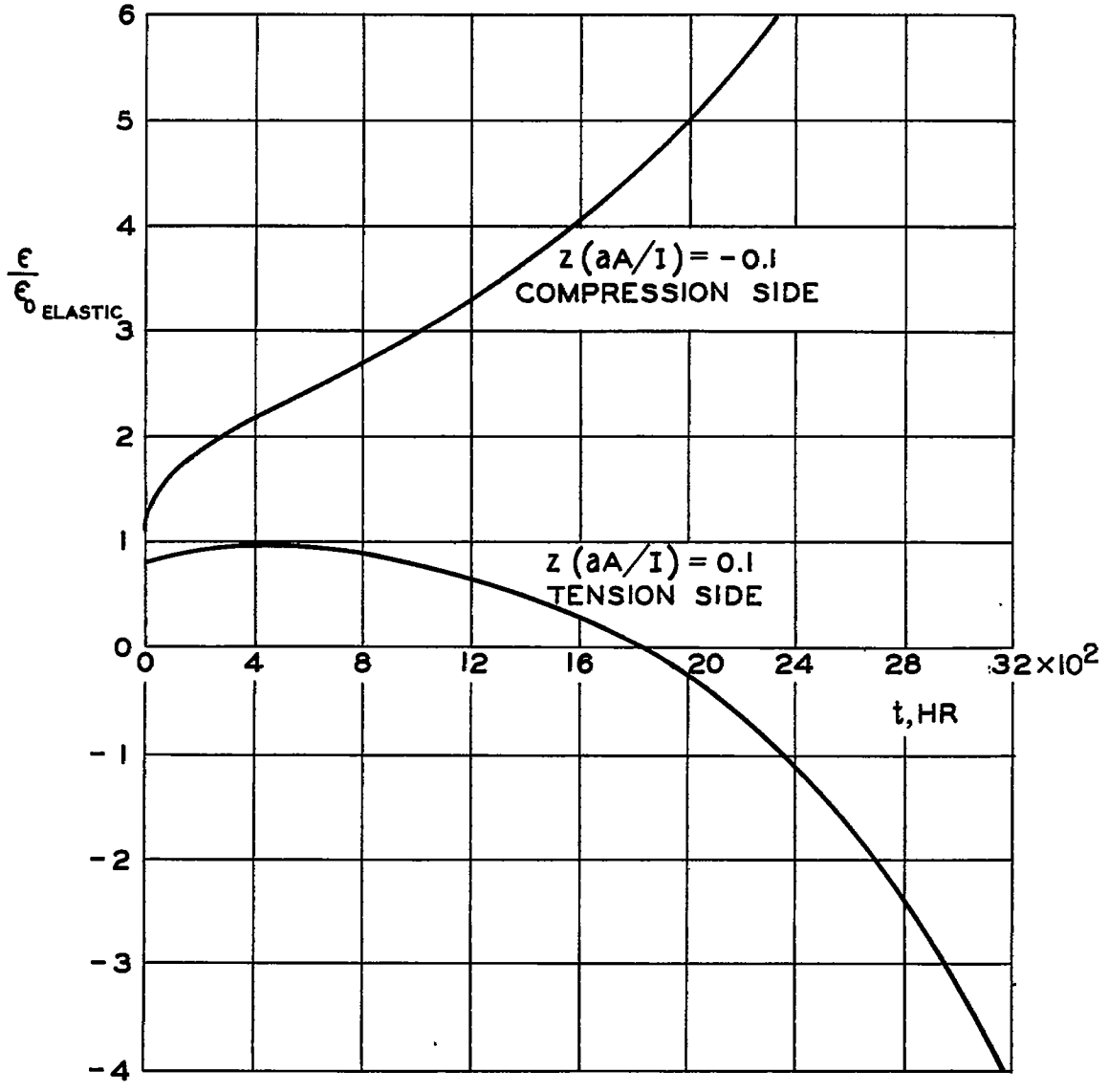

Figure 8.- Creep strain at midspan of a simply supported initially curved column. $P/P_E = 0.5$. (See fig. 7 for material properties.)

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